SPIN STRUCTURES AND SPECTRA OF $\mathbb{Z}_2^k$-MANIFOLDS.

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Abstract. We give necessary and sufficient conditions for the existence of pin$^{\pm}$ and spin structures on Riemannian manifolds with holonomy group $\mathbb{Z}_2^k$. For any $n \geq 4$ (resp. $n \geq 6$) we give examples of pairs of compact manifolds (resp. compact orientable manifolds) M_1, M_2, non homeomorphic to each other, that are Laplace isospectral on functions and on p-forms for any p and such that M_1 admits a pin$^{\pm}$ (resp. spin) structure whereas M_2 does not.

Introduction

Any Riemannian manifold M has naturally associated differential operators of second order, the Laplacian Δ acting on smooth functions and more generally, the p-Laplacian Δ_p acting on smooth p-forms for $0 \leq p \leq n$. The Dirac operator D is a first order operator that can not always be defined. To make this possible, M needs to have an additional structure: a spin structure, if M is orientable, and a pin$^{\pm}$ structure, in general. In this case one says that M is spin or pin$^{\pm}$, respectively.

In this paper we consider a question posed by David Webb, namely, can one hear the property of being spin on a compact Riemannian manifold? We shall answer this question in the negative by giving several examples of Laplace isospectral Riemannian manifolds M_1, M_2 such that M_1 is spin (resp. pin$^{\pm}$) but M_2 has no spin (resp. pin$^{\pm}$) structure. All our examples will be isospectral on p-forms for $0 \leq p \leq n$ and will be given by $\mathbb{Z}_2^k$-*manifolds*, that is, compact Riemannian manifolds with holonomy group $F \simeq \mathbb{Z}_2^k$. We note that by the Cartan-Ambrose-Singer theorem, such a manifold is necessarily flat, hence of the form $M_\Gamma = \Gamma\backslash\mathbb{R}^n$, Γ a Bieberbach group.

In one of the main results, Theorem 2.1, we give a parametrization of the pin$^{\pm}$ or spin structures of M_Γ, showing that the number is either 2^r for some $r \geq k$ or zero, and deriving a simple criterion for non existence (see Remark 2.3). In Section 3 we apply Theorem 2.1 and this criterion to construct several isospectral pairs M, M' of $\mathbb{Z}_2^2$-manifolds of dimensions $n \geq 4$ (resp. $n \geq 6$), such that M admits a pin$^{\pm}$ (resp. spin) structure while M' does not, thus giving a negative answer to Webb's question. By increasing dimensions, we obtain examples of pairs having these same properties and with the extra condition that both M, M' are Kähler (see Remark 3.1).

In the last section we specialize to the case $k = 1$, i.e. of $\mathbb{Z}_2$-manifolds. We show that any such M_Γ has 2^{n-j} pin$^{\pm}$ structures for some $0 \leq j \leq [\frac{n-1}{2}]$,

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with j determined by the $\mathbb{Z}_2$-action. If furthermore M_Γ is of the so called diagonal type and orientable, it turns out that M_Γ admits 2^n spin structures, as in the case of the n-torus (see [**Fr**]).

1. Preliminaries

Bieberbach manifolds. A crystallographic group is a discrete, cocompact subgroup Γ of the isometry group $I(\mathbb{R}^n)$ of $\mathbb{R}^n$. If Γ is torsion-free, then Γ is said to be a *Bieberbach group.* Such a Γ acts properly discontinuously on $\mathbb{R}^n$, thus $M_\Gamma = \Gamma\backslash\mathbb{R}^n$ is a compact flat Riemannian manifold with fundamental group Γ and furthermore, any such manifold arises in this way. Since $I(\mathbb{R}^n) \simeq \mathrm{O}(n) \ltimes \mathbb{R}^n$, any element $\gamma \in I(\mathbb{R}^n)$ decomposes uniquely as $\gamma = BL_b$, with $B \in \mathrm{O}(n)$ and $b \in \mathbb{R}^n$. The translations in Γ form a normal maximal abelian subgroup of finite index L_Λ, Λ a lattice in $\mathbb{R}^n$ which is B-stable for every $BL_b \in \Gamma$. The restriction to Γ of the canonical projection $r : I(\mathbb{R}^n) \to \mathrm{O}(n)$ given by $BL_b \mapsto B$ is a homomorphism with kernel L_Λ and $r(\Gamma)$ is a finite subgroup of $\mathrm{O}(n)$ isomorphic to $F := L_\Lambda\backslash\Gamma$. It is called the *holonomy group* of Γ and gives the linear holonomy group of the Riemannian manifold M_Γ.

A Bieberbach group Γ is said to be of *diagonal type* (see [**MR2**], Definition 1.3) if there exists an orthonormal $\mathbb{Z}$-basis $\{\lambda_1, \ldots, \lambda_n\}$ of the lattice Λ such that for any element $BL_b \in \Gamma$, $B\lambda_i = \pm\lambda_i$ for $1 \le i \le n$. These Bieberbach groups have a rather simple holonomy action, among those with holonomy group $\mathbb{Z}_2^k$. If Γ is of diagonal type, after conjugation of Γ by an isometry, it may be assumed that Λ is the canonical lattice and that b lies in $\frac{1}{2}\Lambda$ for any $\gamma = BL_b \in \Gamma$. Thus, any $\gamma \in \Gamma$ can be written uniquely as $\gamma = BL_{b_o}L_\lambda$, where the coordinates of b_o are 0 or $\frac{1}{2}$ and $\lambda \in \Lambda$ (see [**MR2**], Lemma 1.4).

Pin and spin groups. For a discussion of the material in this subsection we refer to [**LM**], [**Fr2**] or [**GLP**]. Let $Cl^\pm(n)$ denote the Clifford algebras of $\mathbb{R}^n$ endowed with the definite quadratic forms $\mp\|\cdot\|^2$. If $\{e_1, \ldots, e_n\}$ denotes the canonical basis of $\mathbb{R}^n$, then a basis for $Cl^\pm(n)$ is given by the set $\{e_{i_1} \ldots e_{i_k} : 1 \le i_1 < \cdots < i_k \le n\}$. On $Cl^\pm(n)$ one has the relation $vw + wv = \pm 2\langle v, w\rangle$ for any $v, w \in \mathbb{R}^n$, where $\langle\, ,\, \rangle$ denotes the standard inner product. Thus,

$$\begin{aligned} e_ie_j &= -e_je_i && \text{for } i \neq j \text{ for both } Cl^\pm(n), \\ e_i^2 &= \pm 1 && \text{for } 1 \le i \le n \text{ in } Cl^\pm(n). \end{aligned} \tag{1.1}$$

We have compact Lie subgroups, $\mathrm{Pin}^\pm(n)$, of the group of units of $Cl^\pm(n)$, with $\mathrm{Pin}^\pm(n) = \{v_1 \ldots v_h : v_j \in \mathbb{R}^n, \|v_j\| = 1,\ 1 \le j \le h\}$. The connected component of the identity in both cases is isomorphic to $\mathrm{Spin}(n) = \{v_1 \ldots v_h : v_j \in \mathbb{R}^n, \|v_j\| = 1,\ 1 \le j \le h, h \text{ even}\}$, a compact, simply connected Lie group for $n \ge 3$.

Let α be the canonical involution of $Cl^\pm(n)$ given by $\alpha(v_1 \ldots v_h) = (-1)^h v_1 \ldots v_h$. Then, we have Lie group epimorphisms

$$\mu_\pm : \mathrm{Pin}^\pm(n) \to \mathrm{O}(n)$$

with kernel $\{\pm 1\}$, given by $\mu_\pm(v)(x) = \alpha(v)xv^{-1}$ where $v \in \mathrm{Pin}^\pm(n)$ and $x \in \mathbb{R}^n$. If $v \in \mathbb{R}^n$, $\|v\| = 1$, then $\mu_\pm(v)(x) = -vxv^{-1} = \rho_v(x)$ where ρ_v denotes the orthogonal reflection with respect to the hyperplane orthogonal

to v. When restricted to the connected component of the identity, $\mu := \mu_\pm : \mathrm{Spin}(n) \simeq \mathrm{Pin}^\pm(n)_o \to \mathrm{SO}(n)$ give double coverings.

If A_j is a matrix, for $1 \leq j \leq m$ we will abuse notation by denoting by $\mathrm{diag}(A_1, \ldots, A_m)$ the matrix having A_j in the "diagonal" position j.

Let $B(t) = \left[\begin{smallmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{smallmatrix}\right]$ with $t \in \mathbb{R}$ and put

$$\tau(t_1, \ldots, t_m) = \begin{cases} \mathrm{diag}(B(t_1), \ldots, B(t_m)), & \text{if } n = 2m \\ \mathrm{diag}(B(t_1), \ldots, B(t_m), 1), & \text{if } n = 2m+1. \end{cases}$$

We have that $T = \{\tau(t_1, \ldots, t_m) : t_j \in \mathbb{R}\}$ is a maximal torus of $\mathrm{SO}(n)$. A maximal torus of $\mathrm{Spin}(n)$ is given by

$$\tilde{T} = \Big\{ \prod_{j=1}^{m} \big(\cos t_j + \sin t_j \; e_{2j-1} e_{2j} \big) \; : \; t_j \in \mathbb{R} \Big\}.$$

The restriction $\mu : \tilde{T} \to T$ is a 2-fold cover and

$$\mu\big(\prod_{j=1}^{m} (\cos t_j + \sin t_j \; e_{2j-1} e_{2j}) \big) = \tau(2t_1, \ldots, 2t_m). \tag{1.2}$$

Spin structures and pin$^\pm$ structures. If (M, g) is a Riemannian manifold of dimension n, let $\mathrm{B}(M) = \bigcup_{x \in M} \mathrm{B}_x(M)$ be the bundle of frames on M and $\pi : \mathrm{B}(M) \to M$ the canonical projection. That is, for $x \in M$, $\mathrm{B}_x(M)$ is the set of ordered orthonormal bases $(v_1, \ldots, v_n)$ of $T_x(M)$ and $\pi((v_1, \ldots, v_n)) = x$. $\mathrm{B}(M)$ is a principal $\mathrm{O}(n)$-bundle over M and, if M is orientable, the bundle of oriented frames $\mathrm{B}^+(M)$ is a principal $\mathrm{SO}(n)$-bundle. A *pin$^\pm$ structure* on M is a 2-fold cover $p : \tilde{\mathrm{B}}(M) \to \mathrm{B}(M)$ that is equivariant and so that $\tilde{\pi} : \tilde{\mathrm{B}}(M) \to M$ is a principal $\mathrm{Pin}^\pm(n)$-bundle with $\pi \circ p = \tilde{\pi}$. Similarly, a *spin structure* on an orientable manifold M is an equivariant 2-fold cover $p : \tilde{\mathrm{B}}^+(M) \to \mathrm{B}^+(M)$ where $\tilde{\pi} : \tilde{\mathrm{B}}^+(M) \to M$ is a principal $\mathrm{Spin}(n)$-bundle and $\pi \circ p = \tilde{\pi}$.

A manifold in which a spin or a pin$^\pm$ structure has been chosen is called a *spin* or a *pin$^\pm$ manifold*, respectively. Note that if M is orientable, any pin$^\pm$ structure on M defines a spin structure and conversely.

We will be interested on spin and pin$^\pm$ structures on quotients $M_\Gamma = \Gamma \backslash \mathbb{R}^n$, where Γ is a Bieberbach group. If $M = \mathbb{R}^n$, we have that $\mathrm{B}(\mathbb{R}^n) = \mathbb{R}^n \times \mathrm{O}(n)$, thus clearly $\mathbb{R}^n \times \mathrm{Pin}^\pm(n)$ are principal $\mathrm{Pin}^\pm(n)$-bundles and the maps $Id \times \mu_\pm : \mathbb{R}^n \times \mathrm{Pin}^\pm(n) \to \mathbb{R}^n \times \mathrm{O}(n)$ are equivariant 2-fold covering maps. Similarly, we have that $\mathbb{R}^n \times \mathrm{Spin}(n)$ is a principal $\mathrm{Spin}(n)$-bundle and an equivariant 2-fold cover of $\mathrm{B}^+(\mathbb{R}^n) = \mathbb{R}^n \times \mathrm{SO}(n)$. Thus we have spin and pin$^\pm$ structures on $\mathbb{R}^n$ and since $\mathbb{R}^n$ is contractible these are the only such structures. Now, if Γ is a Bieberbach group we have a left action of Γ on $\mathrm{B}(M)$ given by $\gamma \cdot (x, (w_1, \ldots, w_n)) = (\gamma x, (\gamma_* w_1, \ldots, \gamma_* w_n))$. If $\gamma = BL_b$ then $\gamma_* w_j = w_j B$. Fix $(v_1, \ldots, v_n) \in \mathrm{B}(M)$. Since $(w_1, \ldots, w_n) = (v_1 k, \ldots, v_n k)$ for some $k \in \mathrm{O}(n)$, we see that $\gamma_* w_j = (v_j k) B = v_j (Bk)$, thus the action of Γ on $\mathrm{B}(M)$ corresponds to the action of Γ on $\mathbb{R}^n \times \mathrm{O}(n)$ given by $\gamma \cdot (x, k) = (\gamma x, Bk)$.

Now assume that there is a group homomorphism $\varepsilon : \Gamma \to \mathrm{Spin}(n)$ (resp. $\varepsilon_\pm : \Gamma \to \mathrm{Pin}^\pm(n)$) such that $\mu(\varepsilon(\gamma)) = r(\gamma)$ (resp. $\mu_\pm(\varepsilon_\pm(\gamma)) = r(\gamma)$). In this case we can lift the left action of Γ on $\mathrm{B}^+(\mathbb{R}^n)$ (resp. on $\mathrm{B}(\mathbb{R}^n)$) to

$\tilde{\mathrm{B}}^+(\mathbb{R}^n) = \mathbb{R}^n \times \mathrm{Spin}(n)$ (resp. to $\tilde{\mathrm{B}}(\mathbb{R}^n) = \mathbb{R}^n \times \mathrm{Pin}^{\pm}(n)$) via $\gamma \cdot (x, \tilde{k}) = (\gamma x, \varepsilon(\gamma)\tilde{k})$. Thus we have the spin structure

$$\begin{tikzcd} \Gamma\backslash(\mathbb{R}^n \times \mathrm{Spin}(n)) \arrow[rr, "\overline{Id\times\mu}"] \arrow[dr] & & \Gamma\backslash(\mathbb{R}^n \times \mathrm{SO}(n)) \arrow[dl] \\ & \Gamma\backslash\mathbb{R}^n & \end{tikzcd}$$

for M_Γ since $\Gamma\backslash \mathrm{B}(\mathbb{R}^n) = \mathrm{B}(\Gamma\backslash\mathbb{R}^n)$ and $\overline{Id \times \mu}$ is equivariant. Similarly for the pin$^{\pm}$ structures.

In this way, for each homomorphism ε or $\varepsilon_{\pm}$ as above, we obtain a spin or a pin$^{\pm}$ structure on M_Γ, respectively. It turns out that all spin and pin$^{\pm}$ structures on M_Γ are obtained in this manner (see [**Fr2**], [**LM**]).

The n-torus admits 2^n spin structures. Indeed, if $T_\Lambda = \Lambda\backslash\mathbb{R}^n$, and $\lambda_1, \ldots, \lambda_n$ is a $\mathbb{Z}$-basis of Λ, then a homomorphism ε as above is determined by the n-tuple $\varepsilon(L_{\lambda_i}) = \delta_i \in \{\pm 1\}$, for $1 \le i \le n$ (see [**Fr**]). We shall show in Section 4 that this is still the number of such structures for flat manifolds with holonomy group $\mathbb{Z}_2$ which are of diagonal type.

2. Spin and pin$^{\pm}$ structures on $\mathbb{Z}_2^k$-manifolds.

In this section we study the existence of pin$^{\pm}$ structures on $\mathbb{Z}_2^k$-manifolds, showing that the number of such structures is either 0 or 2^r for some $r \ge k$. As an application, in the next section we will construct many examples of $\mathbb{Z}_2^2$-manifolds for any $n \ge 4$, having pin$^+$ structures but no pin$^-$ structures (and conversely) or else, having neither of them.

Let Γ be a Bieberbach group with holonomy group $F \simeq \mathbb{Z}_2^k$, $1 \le k \le n-1$, and translation lattice Λ. Then $M_\Gamma = \Gamma\backslash\mathbb{R}^n$ with $\Gamma = \langle \gamma_1, \ldots, \gamma_k, \Lambda\rangle$ where $\gamma_i = B_i L_{b_i}$, $B_i \in \mathrm{O}(n)$, $b_i \in \mathbb{R}^n$, $B_i\Lambda = \Lambda$, $B_i^2 = Id$ and $B_iB_j = B_jB_i$, for each $1 \le i, j \le k$.

Assume there is a pin$^{\pm}$ structure on M_Γ, that is, a group homomorphism $\varepsilon_{\pm} : \Gamma \to \mathrm{Pin}^{\pm}(n)$ such that $\mu_{\pm} \circ \varepsilon_{\pm} = r$. Then, necessarily $\varepsilon_{\pm}(L_\lambda) \in \{\pm 1\}$, for $\lambda \in \Lambda$. Thus, if $\lambda_1, \ldots, \lambda_n$ is a $\mathbb{Z}$-basis of Λ and we set $\delta_i := \varepsilon_{\pm}(L_{\lambda_i})$, for every $\lambda = \sum_i m_i\lambda_i \in \Lambda$ with $m_i \in \mathbb{Z}$, we have $\varepsilon_{\pm}(L_\lambda) = \prod_i \delta_i^{m_i} = \prod_{m_i \text{odd}} \delta_i$.

If $\gamma = BL_b \in \Gamma$ we will fix a distinguished (though arbitrary) element in $\mu_{\pm}^{-1}(B)$, denoted by $u_{\pm}(B)$. If M_Γ is orientable, we write $u(B) := u_{\pm}(B)$. Thus, if $\gamma = BL_b \in \Gamma$, then

$$\varepsilon_{\pm}(\gamma) = \sigma\, u_{\pm}(B), \tag{2.1}$$

where $\sigma \in \{\pm 1\}$ depends on γ and on the choice of $u_{\pm}(B)$.

Let $\Gamma = \langle \gamma_1, \ldots, \gamma_k, \Lambda\rangle$. The morphism $\varepsilon_{\pm}$ is determined by its action on the generators of Γ, that is, by the $(n+k)$-tuple

$$\begin{aligned} &(\delta_1, \ldots, \delta_n, \sigma_1 u_{\pm}(B_1), \ldots, \sigma_k u_{\pm}(B_k)) \\ \text{or} \qquad &(\delta_1, \ldots, \delta_n, \sigma_1, \ldots, \sigma_k) \in \{\pm 1\}^{n+k} \end{aligned} \tag{2.2}$$

where $\delta_i = \varepsilon_{\pm}(L_{\lambda_i})$ and σ_i is defined by the equation $\varepsilon_{\pm}(\gamma_i) = \sigma_i u_{\pm}(B_i)$, for $1 \le i \le k$.

Now, since $\varepsilon_{\pm}$ is a homomorphism, for any $\gamma = BL_b \in \Gamma, \lambda \in \Lambda$ we have

$$\varepsilon_{\pm}(L_{B\lambda}) = \varepsilon_{\pm}(\gamma L_\lambda \gamma^{-1}) = \varepsilon_{\pm}(\gamma)\varepsilon_{\pm}(L_\lambda)\varepsilon_{\pm}(\gamma^{-1}) = \varepsilon_{\pm}(L_\lambda).$$

Therefore we see that if $\varepsilon_\pm$ is a pin$^\pm$ structure on M_Γ, since $\gamma^2 \in L_\Lambda$, then the character $\varepsilon_{\pm|\Lambda}$ must satisfy the following conditions for any $\gamma = BL_b \in \Gamma$:

$$\begin{array}{lll} & (\varepsilon_1) & \varepsilon_\pm(\gamma^2) = \varepsilon_\pm(\gamma)^2 = u_\pm^2(B) \\ (2.3) & & \\ & (\varepsilon_2) & \varepsilon_\pm(L_{(B-Id)\lambda}) = 1, \quad \lambda \in \Lambda. \end{array}$$

We thus set

$$\hat{\Lambda}(\Gamma) := \{\chi \in \mathrm{Hom}(\Lambda, \{\pm 1\}) \,:\, \chi \text{ satisfies } (\varepsilon_1) \text{ and } (\varepsilon_2)\}. \tag{2.4}$$

The next result gives a parametrization of the pin$^\pm$ structures $\varepsilon_\pm$ for M_Γ.

Theorem 2.1. *If $\Gamma = \langle \gamma_1, \ldots, \gamma_k, \Lambda\rangle$ is a Bieberbach group with holonomy group $\mathbb{Z}_2^k$ and $\sigma_1, \ldots, \sigma_k$ are as in (2.2), then the map $\varepsilon_\pm \mapsto (\varepsilon_{\pm|\Lambda}, \sigma_1, \ldots, \sigma_k)$ defines a bijective correspondence between the pin$^\pm$ structures on M_Γ and the set $\hat{\Lambda}(\Gamma) \times \{\pm 1\}^k$. The number of pin$^\pm$ structures on M_Γ is either* 0 *or* 2^r *for some $r \geq k$.*

Proof. We shall write $\varepsilon, \mu, u(B)$ in place of $\varepsilon_\pm$, $\mu_\pm$, $u_\pm(B)$, for simplicity.

Any element $\gamma \in \Gamma$ can be written as a product of generators $\gamma_i = B_i L_{b_i}$ and $\lambda \in \Lambda$. After reordering, by normality of Λ in Γ and since $B_i^2 = Id$, we see that γ can be written uniquely as

$$\gamma = \gamma_{i_1} \ldots \gamma_{i_r} L_\lambda, \quad \text{ with } 1 \leq i_1 < \cdots < i_r \leq k,\ \lambda \in \Lambda. \tag{2.5}$$

Given $\varepsilon \in \hat{\Lambda}(\Gamma)$ and for any choices of $\varepsilon(\gamma_i) \in \mu^{-1}(B_i)$, $1 \leq i \leq k$, we define (in the notation of (2.5)) for $\gamma \in \Gamma$:

$$\varepsilon(\gamma) = \varepsilon(\gamma_{i_1}) \ldots \varepsilon(\gamma_{i_r})\varepsilon(L_\lambda). \tag{2.6}$$

Thus, we get a well defined map $\varepsilon : \Gamma \to \mathrm{Pin}^\pm(n)$ such that $\mu \circ \varepsilon = r$ and we claim it is a homomorphism. For this purpose we need to show that

$$\varepsilon(\gamma_{i_1} \ldots \gamma_{i_r} L_\lambda \gamma_{j_1} \ldots \gamma_{j_t} L_{\lambda'}) = \varepsilon(\gamma_{i_1} \ldots \gamma_{i_r} L_\lambda)\varepsilon(\gamma_{j_1} \ldots \gamma_{j_t} L_{\lambda'}), \tag{2.7}$$

for any $i_1 < \cdots < i_r$, $j_1 < \cdots < j_t$ and $\lambda, \lambda' \in \Lambda$.

We first note that we may leave out λ, λ' in (2.7). Indeed, assume that for $\gamma, \gamma' \in \Gamma$ one has $\varepsilon(\gamma\gamma') = \varepsilon(\gamma)\varepsilon(\gamma')$. Then, by (ε_2)

$$\begin{aligned} \varepsilon(\gamma L_\lambda \gamma' L_{\lambda'}) &= \varepsilon(\gamma\gamma' L_{B\lambda+\lambda'}) = \varepsilon(\gamma\gamma')\varepsilon(L_{B\lambda+\lambda'}) \\ &= \varepsilon(\gamma)\varepsilon(\gamma')\varepsilon(L_\lambda)\varepsilon(L_{\lambda'}) = \varepsilon(\gamma L_\lambda)\varepsilon(\gamma' L_{\lambda'}). \end{aligned}$$

As a step in the proof of (2.7) (with $\lambda = \lambda' = 0$) we will first show that

$$\varepsilon(\gamma_i\gamma_j) = \varepsilon(\gamma_i)\varepsilon(\gamma_j), \text{ for any } i, j. \tag{2.8}$$

This follows from the definition of ε, if $i < j$, and from condition (ε_1), if $i = j$. We thus assume that $j < i$. Then we may write $\gamma_i\gamma_j = \gamma_j\gamma_i[\gamma_i{}^{-1}, \gamma_j{}^{-1}]$. Since $[\gamma_i{}^{-1}, \gamma_j{}^{-1}] \in \Lambda$, by the definition of ε

$$\varepsilon(\gamma_i\gamma_j) = \varepsilon(\gamma_j)\varepsilon(\gamma_i)\varepsilon([\gamma_i{}^{-1}, \gamma_j{}^{-1}]). \tag{2.9}$$

Note that (2.9) will equal $\varepsilon(\gamma_i)\varepsilon(\gamma_j)$ if and only if it holds the relation

$$\varepsilon([\gamma_i{}^{-1}, \gamma_j{}^{-1}]) = [\varepsilon(\gamma_i{}^{-1}), \varepsilon(\gamma_j{}^{-1})]. \tag{2.10}$$

To show (2.10), we have by condition (ε_1) that

$$\varepsilon((\gamma_j\gamma_i)^2) = \varepsilon(\gamma_j\gamma_i)^2 = \varepsilon(\gamma_j)\varepsilon(\gamma_i)\varepsilon(\gamma_j)\varepsilon(\gamma_i). \tag{2.11}$$

On the other hand

$$\begin{aligned}\varepsilon((\gamma_j\gamma_i)^2) &= \varepsilon(\gamma_j\gamma_j\gamma_i[{\gamma_i}^{-1},{\gamma_j}^{-1}]\gamma_i)\\ &= \varepsilon(\gamma_j^2\gamma_i^2({\gamma_i}^{-1}[{\gamma_i}^{-1},{\gamma_j}^{-1}]\gamma_i))\\ &= \varepsilon({\gamma_j}^2)\varepsilon({\gamma_i}^2)\varepsilon({\gamma_i}^{-1}[{\gamma_i}^{-1},{\gamma_j}^{-1}]\gamma_i)\\ &= \varepsilon(\gamma_j)^2\varepsilon(\gamma_i)^2\varepsilon([{\gamma_i}^{-1},{\gamma_j}^{-1}]).\end{aligned} \tag{2.12}$$

In the last equality we have used condition (ε_2) and the fact that commutators lie in Λ.

Now, by combining (2.11) and (2.12) we obtain (2.10), hence (2.8) follows.

In the general case, (2.7) can be proved by an inductive argument.

Let first $t = 1$, r arbitrary. The case $r = 1$ is (2.8), so assume $r > 1$. If $j_1 > i_r$, then the assertion is clear by the definition of ε, while if $j_1 = i_r$, we may use (ε_1) and induction. We thus assume that there is α such that $i_{\alpha-1} \leq j_1 < i_\alpha$. Actually, we shall take $i_{\alpha-1} < j_1$. The proof when $i_{\alpha-1} = j_1$ is similar, but simpler.

If we set $u = \gamma_{i_\alpha}\cdots\gamma_{i_r}$ then

$$\begin{aligned}\varepsilon(\gamma_{i_1}\cdots\gamma_{i_r}\gamma_{j_1}) &= \varepsilon(\gamma_{i_1}\cdots\gamma_{i_{\alpha-1}}\gamma_{j_1}u[u^{-1},\gamma_{j_1}^{-1}])\\ &= \varepsilon(\gamma_{i_1})\cdots\varepsilon(\gamma_{i_{\alpha-1}})\varepsilon(\gamma_{j_1})\varepsilon(u)\varepsilon([u^{-1},\gamma_{j_1}^{-1}])\\ \text{(by (2.10))} \quad &= \varepsilon(\gamma_{i_1})\cdots\varepsilon(\gamma_{i_{\alpha-1}})\varepsilon(u)\varepsilon(\gamma_{j_1})\\ &= \varepsilon(\gamma_{i_1}\cdots\gamma_{i_r})\varepsilon(\gamma_{j_1}).\end{aligned}$$

The argument for arbitrary t is quite similar and will be omitted. □

Remark 2.2. For manifolds of diagonal type, condition (ε_2) always holds, since $(B - Id)\Lambda \subset 2\Lambda$ for any $BL_b \in \Gamma$. More generally, for manifolds whose holonomy representation decomposes as a sum of integral representations of rank ≤ 2, condition (ε_2) can be expressed in simple terms.

In Section 4 we will study in more detail the case of $\mathbb{Z}_2$-manifolds, showing in particular that pin$^\pm$ structures can always be defined in this case.

Remark 2.3. The previous theorem shows that there are restrictions for a $\mathbb{Z}_2^k$-manifold M_Γ to carry a pin$^\pm$ structure. As a consequence, one has the following simple criterion:

Suppose there exist $\gamma = BL_b, \gamma' = B'L_{b'} \in \Gamma$ with $\gamma^2 = {\gamma'}^2$ and such that for $u_+(B) \in \mu_+^{-1}(B)$ and $u_+(B') \in \mu_+^{-1}(B')$ one has $u_+(B)^2 = -u_+(B')^2$. Then M_Γ can not admit a pin$^+$ structure.

Indeed, such a structure ε_+ would have to satisfy $\varepsilon_+(\gamma) = \pm u_+(B)$, $\varepsilon_+(\gamma') = \pm u_+(B')$ and $\varepsilon_+(\gamma^2) = \varepsilon_+({\gamma'}^2)$, that is, $u_+(B)^2 = u_+(B')^2$ against our assumption.

The same criterion, with the obvious changes, is valid for non existence of pin$^-$ structures, or spin structures in the orientable case.

Remark 2.4. In contrast with Remark 2.3, by applying the doubling procedure in [**DM2**], we may obtain spin Bieberbach manifolds of diagonal type with holonomy group $\mathbb{Z}_2^k$, for any $k \geq 1$. Indeed, let $\Gamma = \langle\gamma_1,\ldots,\gamma_k,L_\Lambda\rangle$ be an n-dimensional Bieberbach group of diagonal type with holonomy group $\mathbb{Z}_2^k$. Define $\mathrm{d}\Gamma := \langle d\gamma_1,\ldots,d\gamma_k,L_{\Lambda\oplus\Lambda}\rangle$ where $d\gamma := \left[\begin{smallmatrix} B & 0\\ 0 & B\end{smallmatrix}\right] L_{(b,b)}$ if $\gamma = BL_b \in \Gamma$ (see Definition 3.1 in [**DM2**]). Thus, $\mathrm{d}\Gamma$ is a Bieberbach group

of dimension $2n$ with holonomy group $\mathbb{Z}_2^k$. The manifold $M_{\mathrm{d}\Gamma} = \mathrm{d}\Gamma\backslash\mathbb{R}^{2n}$ is an orientable Kähler flat manifold of diagonal type. If we apply this procedure twice, then the manifold $M_{\mathrm{d}^2\Gamma}$ is hyperkähler (see Proposition 3.2 in [**DM2**]). It turns out that this $4n$-dimensional manifold is always spin. Indeed, in the notation of Lemma 3.1 in the next section, since $h \in 4\mathbb{Z}$ for $\mathrm{d}^2\Gamma$, we have that $u^2(B) = u_{0,h}^2 = 1$ by (3.3). Hence, condition (ε_1) takes the form $\varepsilon(\gamma^2) = 1$ for any $\gamma \in \Gamma$. Therefore, spin structures can always be defined for $M_{\mathrm{d}^2\Gamma}$, for example we may take any of the 2^k homomorphisms $\varepsilon : \Gamma \to \mathrm{Spin}(n)$ such that $\varepsilon_{|\Lambda} \equiv 1$.

3. Spin structures on some isospectral pairs.

In this section we will construct several isospectral pairs $\{M, M'\}$ of $\mathbb{Z}_2^2$-manifolds of dimension 4 by using the results in [**MR2**], and we will determine the pin$^\pm$ or spin structures, showing that, for some of them, M has a pin$^\pm$ or a spin structure, while M' does not. The main result is given in Theorem 3.2. In the proof, we will need to know some preimages in $\mathrm{Pin}^\pm(n)$ by $\mu_\pm$, as well as their squares.

Set $J := \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. For each $0 \le j, h < n$, we set

$$B_{j,h} = \mathrm{diag}(\underbrace{J, \ldots, J}_{j}, \underbrace{-1, \ldots, -1}_{h}, \underbrace{1, \ldots, 1}_{l}), \tag{3.1}$$

where $n = 2j + h + l$, $j + h \neq 0$ and $l \ge 1$.

Lemma 3.1. *Let $B_{j,h}$ be as in (3.1) and let $\mu_\pm : Pin^\pm(n) \to O(n)$ be the canonical covering maps. If we set*

$$u_{j,h}^\pm := (\tfrac{\sqrt{2}}{2})^j (e_1 - e_2) \ldots (e_{2j-1} - e_{2j})\, e_{2j+1} \cdots e_{2j+h}, \tag{3.2}$$

then $\mu_+^{-1}(B_{j,h}) = \{\pm u_{j,h}^+\}$, $\mu_-^{-1}(B_{j,h}) = \{\pm u_{j,h}^-\}$ and furthermore

$$\begin{aligned} (u_{j,h}^+)^2 &= (-1)^{jh}(-1)^{[\frac{j}{2}]}(-1)^{[\frac{h}{2}]} \\ (u_{j,h}^-)^2 &= (-1)^{jh}(-1)^{[\frac{j+1}{2}]}(-1)^{[\frac{h+1}{2}]}. \end{aligned} \tag{3.3}$$

In particular, $(u_{0,h}^+)^2 = (-1)^{[\frac{h}{2}]}$ and $(u_{0,h}^-)^2 = (-1)^{[\frac{h+1}{2}]}$. If $B_{j,h} \in SO(n)$, i.e. if $j + h$ is even, then $u_{j,h}^2 = (-1)^{\frac{j+h}{2}}$.

If $B \in O(n)$ is conjugate to $B_{j,h}$, and $u_\pm(B) \in \mu_\pm^{-1}(B)$, then $u_\pm^2(B) = (u_{j,h}^\pm)^2$.

Proof. Since $\mu_\pm(e_i) = \rho_{e_i} = \mathrm{diag}(1, \ldots, 1, -1, 1, \ldots, 1)$ with -1 in the i-th position, it is clear that $\mu_+^{-1}(B_{0,h}) = \mu_-^{-1}(B_{0,h}) = \{\pm e_1 \ldots e_h\}$. If $n = 2$, we may write J as a product $J = \left[\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$. Hence, using (1.2), and $\mu_\pm(e_i) = \rho_{e_i}$, we get that $\mu_+^{-1}(J) = \mu_-^{-1}(J) = \{\pm e_1(\cos(\frac{\pi}{4}) + \sin(\frac{\pi}{4}) e_1 e_2)\} = \{\pm \frac{\sqrt{2}}{2}(e_1 - e_2)\}$. Arguing similarly for arbitrary n, the first assertion in the lemma follows.

On the other hand one computes, using (1.1), that both $(e_1 \ldots e_h)^2$ and $2^{-h}((e_1 - e_2) \cdots (e_{2h-1} - e_{2h}))^2$ equal $(-1)^{[\frac{h}{2}]}$ in $Cl^+(n)$ and $(-1)^{[\frac{h+1}{2}]}$ in $Cl^-(n)$, respectively. This implies equations (3.3).

Now, suppose $B = CB_{j,h}C^{-1}$ with $C \in \mathrm{O}(n)$. If $u_+(C) \in \mu_+(C)^{-1}$, then $u_+(B) = \pm u_+(C)u^+_{j,h}u_+(C)^{-1}$ and hence $u^2_+(B) = u_+(C)(u^+_{j,h})^2u_+(C)^{-1} = (u^+_{j,h})^2$. The verification for $u^2_-(B)$ is identical. $\square$

We now consider some pairs of 4-dimensional $\mathbb{Z}_2^2$-manifolds $\{M_i, M_i'\}$, $1 \le i \le 5$, where $M_i = \Gamma_i \backslash \mathbb{R}^4$, $M_i' = \Gamma_i' \backslash \mathbb{R}^4$ and the groups $\Gamma_i = \langle \gamma_1, \gamma_2, \Lambda \rangle$, $\Gamma_i' = \langle \gamma_1', \gamma_2', \Lambda \rangle$ are given in Table 1, where $\gamma_i = B_i L_{b_i}$, $\gamma_i' = B_i L_{b_i'}$, $i = 1, 2$, $B_3 = B_1 B_2$, $b_3 = B_2 b_1 + b_2$, $b_3' = B_2' b_1' + b_2'$ and $\Lambda = \mathbb{Z}e_1 \oplus \ldots \oplus \mathbb{Z}e_n$ is the canonical lattice. Furthermore, we take $B_i = B_i'$. In all cases the matrices B_i are diagonal and are written as column vectors. We indicate the translation vectors b_i, b_i' also as column vectors, leaving out the coordinates that are equal to zero. We will also use the pair $\{\tilde{M}_1, \tilde{M}_1'\}$ of $\mathbb{Z}_2^2$-manifolds of dimension 6 obtained from the pair $\{M_1, M_1'\}$ by adjoining the characters $(-1, 1, -1)$ and $(1, -1, -1)$ to B_i, $1 \le i \le 3$, and keeping b_i, b_i' unchanged.

Table 1

	B_1	L_{b_1}	$L_{b_1'}$	B_2	L_{b_2}	$L_{b_2'}$	B_3	L_{b_3}	$L_{b_3'}$
	1			1	1/2	1/2	1	1/2	1/2
$\{M_1, M_1'\}$	1		1/2	1	1/2		1	1/2	1/2
$\{\tilde{M}_1, \tilde{M}_1'\}$	1	1/2		-1			-1	1/2	
	-1			1		1/2	-1		1/2
	-1			1			-1		
	1			-1			-1		

	B_1	L_{b_1}	$L_{b_1'}$	B_2	L_{b_2}	$L_{b_2'}$	B_3	L_{b_3}	$L_{b_3'}$
	1			1		1/2	1		1/2
$\{M_2, M_2'\}$	1		1/2	1	1/2	1/2	1	1/2	
	1	1/2		-1			-1	1/2	
	-1			1	1/2		-1	1/2	

	B_1	L_{b_1}	$L_{b_1'}$	B_2	L_{b_2}	$L_{b_2'}$	B_3	L_{b_3}	$L_{b_3'}$
	1			-1			-1		
$\{M_3, M_3'\}$	1		1/2	-1			-1		1/2
	-1			-1	1/2		1	1/2	
	1	1/2		1	1/2	1/2	1		1/2

	B_1	L_{b_1}	$L_{b_1'}$	B_2	L_{b_2}	$L_{b_2'}$	B_3	L_{b_3}	$L_{b_3'}$
	1	1/2		-1			-1	1/2	
$\{M_4, M_4'\}$	1	1/2	1/2	-1			-1	1/2	1/2
	-1			-1		1/2	1		1/2
	1		1/2	1	1/2	1/2	1	1/2	

	B_1	L_{b_1}	$L_{b_1'}$	B_2	L_{b_2}	$L_{b_2'}$	B_3	L_{b_3}	$L_{b_3'}$
	-1			1		1/2	-1		1/2
$\{M_5, M_5'\}$	-1			-1	1/2	1/2	1	1/2	1/2
	1		1/2	-1			-1		1/2
	1	1/2		1	1/2		1		

We observe that only $M_5, M_5', \tilde{M}_1, \tilde{M}_1'$ are orientable.

In order to show the isospectrality of these pairs we will need to recall some known results.

For $BL_b \in \Gamma$ set $n_B := \dim(\mathbb{R}^n)^B = |\{1 \le i \le n : Be_i = e_i\}|$ and

$$n_B(\tfrac{1}{2}) := |\{1 \le i \le n : Be_i = e_i \text{ and } b \cdot e_i = \tfrac{1}{2}\}|. \tag{3.4}$$

If $0 \le t \le d \le n$, the *Sunada numbers* for Γ are defined by

$$c_{d,t}(\Gamma) := \left|\left\{BL_b \in \Gamma : n_B = d \text{ and } n_B(\tfrac{1}{2}) = t\right\}\right|. \tag{3.5}$$

In [**MR2**], Theorem 3.3, it is shown that the equality of the Sunada numbers $c_{d,t}(\Gamma) = c_{d,t}(\Gamma')$ for every d, t, is equivalent to the validity of the conditions in Sunada's theorem (see [**Su**]) for M_Γ and $M_{\Gamma'}$. In particular this implies that M_Γ and $M_{\Gamma'}$ are isospectral on p-forms for $0 \le p \le n$. This method was used in [**MR**] and [**MR3**] to prove the isospectrality of the pairs M_5, M_5' and M_2, M_2' respectively. Also, the method of adding characters and keeping isospectrality was also used in [**MR**].

We are now in a position to state the main result in this paper.

Theorem 3.2. *The pairs M_i, M_i', $1 \le i \le 5$, and $\tilde{M}_1, \tilde{M}_1'$ are pairwise isospectral.*

The number of pin$^\pm$ and spin structures on M_i, M_i', $1 \le i \le 5$, and $\tilde{M}_1, \tilde{M}_1'$ are given in the following table.

Pairs	M_1	M_1'	$\tilde{M}_1$	$\tilde{M}_1'$	M_2	M_2'	M_3	M_3'	M_4	M_4'	M_5	M_5'
pin$^+$	–	2^3	–	2^5	2^3	–	–	2^4	2^4	2^3	2^4	2^3
pin$^-$	2^4	2^3	–	2^5	2^3	–	–	–	–	2^3	2^4	2^3
spin	–	–	–	2^5	–	–	–	–	–	–	2^4	2^3

The various isospectral pairs in the table show that one can not hear the existence of pin$^\pm$ or spin structures on a compact Riemannian manifold.

Proof. Since all manifolds are of diagonal type, to show that these pairs are isospectral it suffices to check the equality of the Sunada numbers (see (3.5)). It is easy to see from Table 1 that the non trivial Sunada numbers, besides $c_{4,0} = 1$ corresponding to the identity, are: $c_{2,2} = c_{3,1} = c_{3,2} = 1$ for M_1 and M_1'; $c_{2,2} = c_{4,1} = c_{4,2} = 1$ for $\tilde{M}_1$ and $\tilde{M}_1'$; $c_{2,1} = c_{3,1} = c_{3,2} = 1$ for M_2 and M_2'; $c_{1,1} = c_{2,1} = c_{3,1} = 1$ for M_3 and M_3'; $c_{1,1} = c_{2,1} = c_{3,2} = 1$ for M_4 and M_4'; and $c_{2,1} = 3$ for M_5 and M_5'. Thus, it follows that all pairs $M_i, {M_i}'$, $1 \le i \le 5$, and $\tilde{M}_1, \tilde{M}_1'$ are isospectral on functions.

We shall now use Theorem 2.1 to determine the spin and pin$^\pm$ structures on $M_1, M_1', \dots, M_5, M_5', \tilde{M}_1$ and $\tilde{M}_1'$. By Remark 2.2 we need only look at condition (ε_1).

We first look at the pair M_1, M_1'. We have that

$$\gamma_1^2 = L_{e_3},\ \gamma_2^2 = L_{e_1+e_2} = \gamma_3^2; \qquad {\gamma_1'}^2 = L_{e_2},\ {\gamma_2'}^2 = L_{e_1+e_4},\ {\gamma_3'}^2 = L_{e_1+e_2}.$$

By (1.2) and Lemma 3.1:

$$u_\pm^2(B_1) = u_\pm^2(B_1') = (\sigma_1 e_4)^2 = \pm 1, \qquad u_\pm^2(B_2) = u_\pm^2(B_2') = (\sigma_2 e_3)^2 = \pm 1,$$
$$u_\pm^2(B_3) = u_\pm^2(B_3') = (\sigma_3 e_3 e_4)^2 = -1$$

with $\sigma_i \in \{\pm 1\}$. By the criterion in Remark 2.3, it follows that M_1 has no pin$^+$ structures, since $\gamma_2^2 = \gamma_3^2$ and $u_+^2(B_2) = 1$ while $u_+^2(B_3) = -1$.

Furthermore, by the previous equations, if $\delta_i = \varepsilon_\pm(L_{e_i})$, condition (ε_1) gives $\delta_3 = \pm 1$, $\delta_1\delta_2 = \pm 1$ and $\delta_1\delta_2 = -1$. The last two equations are not compatible for $Cl^+(n)$, hence we see again that M_1 does not admit pin$^+$ structures. However, it has 2^4 pin$^-$ structures given by

$$\varepsilon_-(M_1) = (\delta_1, -\delta_1, -1, \delta_4; \sigma_1 e_4, \sigma_2 e_3)$$

where $\delta_i, \sigma_j \in \{\pm 1\}$ are arbitrary for $i = 1, 4$, $j = 1, 2$. Similarly, condition (ε_1) for M_1' gives $\delta_2 = \pm 1$, $\delta_1\delta_4 = \pm$ and $\delta_1\delta_2 = -1$. Thus, M_1' has 2^3 pin$^\pm$ structures given by

$$\varepsilon_\pm(M_1') = (\mp 1, \pm 1, \delta_3, -1; \sigma_1 e_4, \sigma_2 e_3)$$

with $\delta_3, \sigma_1, \sigma_2 \in \{\pm 1\}$. In this way, we have shown that M_1, M_1' is an isospectral pair such that M_1 carries no pin$^+$ structure while ${M_1}'$ admits 2^3 of them.

We note that the orientable manifolds $\tilde{M}_1, \tilde{M}_1'$ do have the same properties. These manifolds are still isospectral (again we have equality of Sunada numbers) and γ_i^2 and ${\gamma_i'}^2$ are the same as before, for $1 \le i \le 3$.

Now, if we look for spin structures ε on $\tilde{M}_1, \tilde{M}_1'$, we get

$$u^2(B_1) = u^2(B_1') = (\sigma_1 e_4 e_5)^2 = -1, \quad u^2(B_2) = u^2(B_2') = (\sigma_2 e_3 e_6)^2 = -1,$$
$$u^2(B_3) = u^2(B_3') = (\sigma_3 e_3 e_4 e_5 e_6)^2 = 1.$$

For $\tilde{M}_1$ we have $\gamma_2^2 = \gamma_3^2 = L_{e_1+e_2}$, hence $\varepsilon(\gamma_2^2) = \varepsilon(\gamma_3^2)$, a contradiction, given that $u^2(B_2) = -1$ and $u^2(B_3) = 1$. Thus, there are no spin structures on $\tilde{M}_1$. On the other hand, for $\tilde{M}_1'$, we have ${\gamma_1'}^2 = L_{e_2}$, ${\gamma_2'}^2 = L_{e_1+e_4}$, ${\gamma_3'}^2 = L_{e_1+e_2}$. Thus, $\varepsilon(L_{e_2}) = -1$, $\varepsilon(L_{e_1+e_4}) = -1$, $\varepsilon(L_{e_1+e_2}) = 1$, hence there are 2^5 spin structures given by

$$\varepsilon = (-1, -1, \delta_3, 1, \delta_5, \delta_6; \sigma_1 e_4 e_5, \sigma_2 e_3 e_6)$$

with $\delta_3, \delta_5, \delta_6, \sigma_1, \sigma_2 \in \{\pm 1\}$.

This proves the claim and shows that one can not hear the existence of spin structures on a compact Riemannian manifold.

We consider next the remaining pairs M_i, M_i', $2 \le i \le 5$. The calculations are entirely similar to those in the cases discussed above, so we will omit the details, giving the necessary information in several tables. For convenience, we will also include the pair M_1, M_1'.

Note that the manifolds M_1, M_1', M_2, M_2', as well as M_3, M_3', M_4, M_4', have the same holonomy representation. Furthermore, all matrices appearing in Table 1 are conjugate to $B_{0,1}$, $B_{0,2}$ or $B_{0,3}$. By Lemma 3.1 we know that ${u_{0,1}^\pm}^2 = \pm 1$, ${u_{0,2}^\pm}^2 = -1$ and ${u_{0,3}^\pm}^2 = \mp 1$ for $\mathrm{Pin}^\pm(n)$. Thus we have:

TABLE 2.

manifolds	$u_\pm^2(B_1)$	$u_\pm^2(B_2)$	$u_\pm^2(B_3)$
M_1, M_1', M_2, M_2'	± 1	± 1	-1
M_3, M_3', M_4, M_4'	± 1	∓ 1	-1
M_5, M_5'	-1	-1	-1

One has that $\gamma_i^2 = L_{\lambda_i} \in \Lambda$. In Table 3 we give the vectors λ_i for $1 \le i \le 3$ and for every M_j, M_j', $1 \le j \le 5$.

TABLE 3.

	M_1	M_1'	M_2	M_2'	M_3	M_3'	M_4	M_4'	M_5	M_5'
γ_1^2	e_3	e_2	e_3	e_2	e_4	e_2	e_1+e_2	e_2+e_4	e_4	e_3
γ_2^2	e_1+e_2	e_1+e_4	e_2+e_4	e_1+e_2	e_4	e_4	e_4	e_4	e_4	e_1
γ_3^2	e_1+e_2	e_1+e_2	e_2	e_1	e_3	e_4	e_4	e_3	e_2	e_2

Using the information obtained in Tables 2 and 3 we get the equations to be satisfied by the δ_i's, resulting from condition (ε_1).

TABLE 4. Equations for δ_i, $1 \le i \le 4$.

	γ_1	γ_2	γ_3
M_1	$\delta_3 = \pm 1$	$\delta_1\delta_2 = \pm 1$	$\delta_1\delta_2 = -1$
M_1'	$\delta_2 = \pm 1$	$\delta_1\delta_4 = \pm 1$	$\delta_1\delta_2 = -1$
M_2	$\delta_3 = \pm 1$	$\delta_2\delta_4 = \pm 1$	$\delta_2 = -1$
M_2'	$\delta_2 = \pm 1$	$\delta_1\delta_2 = \pm 1$	$\delta_1 = -1$
M_3	$\delta_4 = \pm 1$	$\delta_4 = \mp 1$	$\delta_3 = -1$
M_3'	$\delta_2 = \pm 1$	$\delta_4 = \mp 1$	$\delta_4 = -1$
M_4	$\delta_1\delta_2 = \pm 1$	$\delta_4 = \mp 1$	$\delta_4 = -1$
M_4'	$\delta_2\delta_4 = \pm 1$	$\delta_4 = \mp 1$	$\delta_3 = -1$
M_5	$\delta_4 = -1$	$\delta_4 = -1$	$\delta_2 = -1$
M_5'	$\delta_3 = -1$	$\delta_4 = -1$	$\delta_2 = -1$

By looking at Table 4 we immediately see that M_1 has no pin$^+$ structures, M_2' and M_3 admit no pin$^\pm$ structures and M_3' has no pin$^-$ structures, since the corresponding equations are not compatible. We now list all the characters $\varepsilon_{\pm|\Lambda}$, corresponding to the pin$^\pm$ and spin structures in the remaining cases.

$$\begin{aligned}
\varepsilon_-(M_1) &= (\delta_1, -\delta_1, -1, \delta_4), & \varepsilon_\pm(M_1') &= (\mp 1, \pm 1, \delta_3, -1),\\
\varepsilon_\pm(M_2) &= (\delta_1, -1, \pm 1, \mp 1), & \varepsilon_-(M_3) &= (\delta_1, -1, \delta_3, -1),\\
\varepsilon_+(M_4) &= (\delta_1, \delta_1, \delta_3, -1), & \varepsilon_\pm(M_4') &= (\delta_1, -1, -1, \mp 1),\\
\varepsilon(M_5) &= (\delta_1, -1, \delta_3, -1), & \varepsilon(M_5') &= (-1, -1, -1, \delta_4).
\end{aligned}$$

Now, for each choice of $\varepsilon_{\pm|\Lambda}$ there are $2^2 = 4$ structures corresponding to the possible choices of σ_1, σ_2, hence it is easy to verify that the number of pin$^+$, pin$^-$ or spin structures is as indicated in the theorem. $\square$

Remark 3.3. The procedure of adding appropriate characters to M_1, M_1' to obtain orientable manifolds, with M_1 admitting a spin structure while M_1' does not, can be used with the remaining pairs M_i, M_i', $2 \le i \le 4$, as well. Alternately, we can also use the method described in Remark 2.4. Indeed, consider the orientable $\mathbb{Z}_2^2$-manifolds $M_{\mathrm{d}\Gamma_1}, M_{\mathrm{d}\Gamma_1'}$ of dimension 8 obtained by doubling the Bieberbach groups Γ_1, Γ_1' (see Table 5). The resulting manifolds now carry a Kähler structure.

Table 5.

B_1	L_{b_1}	$L_{b'_1}$	B_2	L_{b_2}	$L_{b'_2}$	B_3	L_{b_3}	$L_{b'_3}$
1			1	1/2	1/2	1	1/2	1/2
1		1/2	1	1/2		1	1/2	1/2
1	1/2		-1			-1	1/2	
-1			1		1/2	-1		1/2
1			1	1/2	1/2	1	1/2	1/2
1		1/2	1	1/2		1	1/2	1/2
1	1/2		-1			-1	1/2	
-1			1		1/2	-1		1/2

By comparing the Sunada numbers, we see that $M_{\mathrm{d}\Gamma_1}$ and $M_{\mathrm{d}\Gamma'_1}$ are isospectral. Now, we look at condition (ε_1) in (2.3). For $\mathrm{d}\Gamma_1$ we have that $\delta_3\delta_7 = -1$, $\delta_1\delta_2\delta_5\delta_6 = -1$ and $\delta_1\delta_2\delta_5\delta_6 = 1$. These last two equations are clearly not compatible, hence $M_{\mathrm{d}\Gamma_1}$ admits no spin structures. On the other hand, for $\mathrm{d}\Gamma'_1$ we get $\delta_2\delta_6 = -1$, $\delta_1\delta_4\delta_5\delta_8 = -1$ and $\delta_1\delta_2\delta_5\delta_6 = 1$, hence $\varepsilon = (\delta_1, \delta_2, \delta_3, \delta_4, -\delta_1, -\delta_2, \delta_7, -\delta_4, \sigma_1 e_4 e_8, \sigma_2 e_3 e_7)$, thus obtaining 2^7 spin structures in this case.

4. Pin$^{\pm}$ structures on $\mathbb{Z}_2$-manifolds.

In this last section we study in some detail the special case of $\mathbb{Z}_2$-manifolds, where an explicit description of the pin$^{\pm}$ structures can be given. For each $0 \le j, h < n$, let as in (3.1)

$$B_{j,h} := \operatorname{diag}(\underbrace{J, \dots, J}_{j}, \underbrace{-1, \dots, -1}_{h}, \underbrace{1, \dots, 1}_{l})$$

where $n = 2j + h + l$, $j + h \neq 0$ and $l \ge 1$. Then $B_{j,h} \in \mathrm{O}(n)$, $B_{j,h}^2 = Id$ and $B_{j,h} \in \mathrm{SO}(n)$ if and only if $j + h$ is even. Let $\Lambda = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n$ be the canonical lattice of $\mathbb{R}^n$ and for j, h as before define the groups

$$\Gamma_{j,h} := \langle B_{j,h} L_{\frac{e_n}{2}}, \Lambda \rangle. \tag{4.1}$$

We have that Λ is stable by $B_{j,h}$ and $(B_{j,h} + Id)\frac{e_n}{2} = e_n \in \Lambda \smallsetminus (B_{j,h} + Id)\Lambda$. Hence, by Proposition 2.1 in [**DM**], the $\Gamma_{j,h}$ are Bieberbach groups. In this way, if $M_{j,h} = \Gamma_{j,h} \backslash \mathbb{R}^n$, we have a family

$$\mathcal{F} = \{ M_{j,h} \,:\, 0 \le j \le [\tfrac{n-1}{2}], 0 \le h < n - 2j,\ j + h \neq 0 \} \tag{4.2}$$

of compact flat manifolds with holonomy group $F \simeq \mathbb{Z}_2$. The next proposition summarizes some known results on $\mathbb{Z}_2$-manifolds. We include a proof for completeness.

Proposition 4.1. *The family $\mathcal{F}$ gives a system of representatives for the diffeomorphism classes of $\mathbb{Z}_2$-manifolds of dimension n. Furthermore we have:*

$$H_1(M_{j,h}, \mathbb{Z}) \simeq \mathbb{Z}^{j+l} \oplus \mathbb{Z}_2^h. \tag{4.3}$$

For $1 \le p \le n$,

$$\beta_p(M_{j,h}) = \sum_{i=0}^{[\frac{p}{2}]} \binom{j+h}{2i}\binom{j+l}{p-2i}. \tag{4.4}$$

If $\beta_1(M_{j,h}) = \beta_1(M_{j',h'})$*, then* $\beta_p(M_{j,h}) = \beta_p(M_{j',h'})$ *for any* $p \ge 1$.

Proof. We first prove that the manifolds $M_{j,h}$ are pairwise non homeomorphic. We now compute $H_1(M_{j,h}, \mathbb{Z}) \simeq \Gamma_{j,h}/[\Gamma_{j,h}, \Gamma_{j,h}]$. For $\gamma = B_{j,h}L_{\frac{e_n}{2}}$, we have

$$\begin{aligned}
[\Gamma_{j,h}, \Gamma_{j,h}] &= \langle [\gamma, L_{e_i}] = L_{(B-Id)e_i} : 1 \le i \le n\rangle \\
&= \langle L_{e_2-e_1}, \ldots, L_{e_{2j}-e_{2j-1}}, L_{2e_{2j+1}}, \ldots, L_{2e_{2j+h}}\rangle.
\end{aligned}$$

Using this information and the fact that $\gamma^2 = L_{e_n}$ it is easy to see that

$$H_1(M_{j,h}, \mathbb{Z}) \simeq \mathbb{Z}^{j+l} \oplus \mathbb{Z}_2^h.$$

Thus, if $M_{j,h}$ and $M_{j',h'}$ are homeomorphic then $h = h'$ and $j + l = j' + l'$, hence $j = j'$ as asserted.

To show that the family $\mathcal{F}$ gives a complete system of representatives for the diffeomorphism classes of $\mathbb{Z}_2$-manifolds, we will use results in [**Ch**], p.153 (it could also be proved directly by using that any integral representation of $\mathbb{Z}_2$ decomposes uniquely as a sum of indecomposable representations of rank ≤ 2 given by $1, -1$ or J).

The cardinality of $\mathcal{F}$ equals $\left(\sum_{j=0}^{[\frac{n-1}{2}]} n - 2j\right) - 1$, since we must exclude the case $j = h = 0$ corresponding to $B_{0,0} = Id$. Thus we have

$$\#\mathcal{F} = \left(n - [\tfrac{n-1}{2}]\right)\left([\tfrac{n-1}{2}] + 1\right) - 1 = \begin{cases} \frac{n^2+2n-4}{4} & n \text{ even} \\ \frac{n^2+2n-3}{4} & n \text{ odd.} \end{cases} \tag{4.5}$$

On the other hand, if p is a prime, Charlap gives a formula for the number N_p of diffeomorphism classes of $\mathbb{Z}_p$-manifolds of dimension n. For $p = 2$ this number is given by:

$$N_2 = \tfrac{1}{2}[\tfrac{n-1}{2}]\Big([\tfrac{n-1}{2}] + 3\Big) + \tfrac{1}{2}\Big((n-1) - [\tfrac{n-1}{2}]\Big)\Big(n - [\tfrac{n-1}{2}]\Big).$$

In this way we obtain that $N_2 = \frac{1}{8}(n-2)(n+4) + \frac{1}{8}n(n+2) = \frac{n^2+2n-4}{4}$ for n even, and $N_2 = \frac{1}{4}(n-1)(n+3) = \frac{n^2+2n-3}{4}$, for n odd. This shows that $\#\mathcal{F} = N_2$, as claimed.

To determine the p-Betti number of $M_{j,h}$ for $1 \le p \le n$, we note that $B_{j,h}$ acts diagonally on the basis $e_1 \pm e_2, \ldots, e_{2j-1} \pm e_{2j}, e_{2j+1}, \ldots, e_n$, with $j + l$ (resp. $j + h$) eigenvectors with eigenvalue 1 (resp. -1). Thus, an exterior product of p elements of this basis will be invariant by $B_{j,h}$, if and only if an even number of them have eigenvalue -1. Hence we have

$$\beta_p(M_{j,h}) = \sum_{i=0}^{[\frac{p}{2}]} \binom{j+h}{2i}\binom{j+l}{p-2i}$$

as asserted. Now, if $\beta_1(M_{j,h}) = \beta_1(M_{j',h'})$ then $j + l = j' + l'$ and hence $j + h = j' + h'$. Thus, $\beta_p(M_{j,h}) = \beta_p(M_{j',h'})$, for any $1 \le p \le n$. $\square$

The next result gives a description of pin$^\pm$ structures on $\mathbb{Z}_2$-manifolds.

Proposition 4.2. *Every $\mathbb{Z}_2$-manifold M_Γ has pin$^\pm$ structures (and spin structures, if M_Γ is orientable). If $\Gamma = \Gamma_{j,h}$ then M_Γ has 2^{n-j} pin$^\pm$ structures parametrized by the tuples $(\delta_1, \ldots, \delta_n, \sigma) \in \{\pm 1\}^{n+1}$ satisfying:*

$$\delta_1 = \delta_2, \ \cdots, \ \delta_{2j-1} = \delta_{2j} \tag{4.6}$$

and

$$\delta_n = \begin{cases} (-1)^{jh}(-1)^{[\frac{j}{2}]}(-1)^{[\frac{h}{2}]} & \textit{for pin}^+ \textit{ structures} \\ (-1)^{jh}(-1)^{[\frac{j+1}{2}]}(-1)^{[\frac{h+1}{2}]} & \textit{for pin}^- \textit{ structures.} \end{cases} \tag{4.7}$$

In particular, in the case of spin structures we have $\delta_n = (-1)^{\frac{j+h}{2}}$.

Proof. In light of Proposition 4.1, we have that $\Gamma \simeq \Gamma_{j,h}$ for some j, h, hence M_Γ is diffeomorphic to $M_{j,h}$. Therefore, since pin$^\pm$ structures on diffeomorphic manifolds are in a bijective correspondence, we may assume that $\Gamma = \Gamma_{j,h}$.

We have observed in Remark 2.2 that equation (ε_2) always holds for $\mathbb{Z}_2$-manifolds of diagonal type. However, in the non diagonal case, (ε_2) gives a restriction. Namely, let $\lambda = \sum_{i=1}^n m_i e_i$, $m_i \in \mathbb{Z}$. Then

$$(B_{j,h} - Id)\lambda = \sum_{i=1}^{j} (m_{2i} - m_{2i-1})e_{2i-1} + (m_{2i-1} - m_{2i})e_{2i} - 2\sum_{i=1}^{h} m_{2j+i}e_{2j+i}.$$

Thus, (ε_2) holds if and only if

$$\delta_1^{(m_2-m_1)}\delta_2^{(m_1-m_2)} \cdots \delta_{2j-1}^{(m_{2j}-m_{2j-1})}\delta_{2j}^{(m_{2j-1}-m_{2j})} = 1$$

for every $m_1, \ldots, m_{2j} \in \mathbb{Z}$, or equivalently,

$$\delta_1 = \delta_2, \ \ldots, \ \delta_{2j-1} = \delta_{2j}.$$

Each of these relations divides by 2 the number of structures. Hence we obtain a maximum of 2^{n-j+1} pin$^\pm$ structures for $M_{j,h}$. Furthermore, equation (ε_1) gives another restriction since $\varepsilon_\pm(\gamma^2) = \varepsilon_\pm(L_{(B+Id)b}) = \varepsilon_\pm(\gamma)^2$. Now $(B + Id)b = e_n$, hence, by (3.3), equation (ε_1) reads:

$$\delta_n = \begin{cases} (-1)^{jh}(-1)^{[\frac{j}{2}]}(-1)^{[\frac{h}{2}]} & \text{in } Cl^+(n) \\ (-1)^{jh}(-1)^{[\frac{j+1}{2}]}(-1)^{[\frac{h+1}{2}]} & \text{in } Cl^-(n). \end{cases} \tag{4.8}$$

Thus, the restriction imposed by (4.8) divides by 2 the number of structures and we get a total of 2^{n-j} pin$^\pm$ structures on M_Γ for $\Gamma = \Gamma_{j,h}$. □

Note. Proposition 4.1 together with Lemma 3.1, give an explicit description of all pin$^\pm$ structures on $\mathbb{Z}_2$-manifolds.

Example 4.3. As a final task, to illustrate Proposition 4.2, we list explicitly the 28 pin$^\pm$ Riemannian $\mathbb{Z}_2$-manifolds (M, ε) of dimension 3 having canonical lattice of translations Λ.

There are 3 diffeomorphism classes, one of which splits into 2 isometry classes, hence we have 4 isometry classes, corresponding to the groups $\Gamma_{1,0} = \left\langle \begin{bmatrix} J & \\ & 1 \end{bmatrix} L_{\frac{e_3}{2}}, \Lambda \right\rangle$, $\Gamma_{0,1} = \left\langle \begin{bmatrix} -1 & & \\ & 1 & \\ & & 1 \end{bmatrix} L_{\frac{e_3}{2}}, \Lambda \right\rangle$, $\Gamma'_{0,1} = \left\langle \begin{bmatrix} -1 & & \\ & 1 & \\ & & 1 \end{bmatrix} L_{\frac{e_2+e_3}{2}}, \Lambda \right\rangle$ and $\Gamma_{0,2} = \left\langle \begin{bmatrix} -1 & & \\ & -1 & \\ & & 1 \end{bmatrix} L_{\frac{e_3}{2}}, \Lambda \right\rangle$.

We note that $M_{0,1}$ and $M'_{0,1}$ are not isometric, as can be seen by computing the injectivity radius, that is the length of the shortest closed geodesic. Indeed, using the results in [**MR3**] one easily sees that these equal $\frac{1}{2}$ and $\frac{\sqrt{2}}{2}$, respectively.

The pairs (M, ε) are listed in the following table, obtained by using Lemma 3.1 and Theorem 2.1.

TABLE 6. $\mathrm{Pin}^{\pm}$ structures on $\mathbb{Z}_2$-manifolds of dimension 3.

M_Γ	cond. (ε_1)	cond. (ε_2)	pin$^{\pm}$ structures	#
$M_{1,0}$	$\delta_3 = \pm 1$	$\delta_1 = \delta_2$	$\varepsilon_{\pm} = (\delta_1, \delta_1, \pm 1; \sigma \frac{\sqrt{2}}{2}(e_1 - e_2))$	2^2
$M_{0,1}$	$\delta_3 = \pm 1$	–	$\varepsilon_{\pm} = (\delta_1, \delta_2, \pm 1; \sigma e_1)$	2^3
$M'_{0,1}$	$\delta_2\delta_3 = \pm 1$	–	$\varepsilon_{\pm} = (\delta_1, \delta_2, \pm \delta_2; \sigma e_1)$	2^3
$M_{0,2}$	$\delta_3 = -1$	–	$\varepsilon_{\pm} = (\delta_1, \delta_2, -1; \sigma e_1 e_2)$	2^3

We note that the spin structures for $M_{0,2}$ are already contained in [**Pf**].

References

[**Ch**] Charlap L., *Bieberbach groups and flat manifolds*, Springer Verlag, Universitext, 1988.

[**DM**] Dotti I., Miatello R., *Isospectral compact flat manifolds*, Duke Math. J. **68** (489–498), 1992.

[**DM2**] Dotti I., Miatello R., *Quaternion Kähler flat manifolds*, Diff. Geom. Appl. **15** (59–77), 2001.

[**Fr**] Friedrich T., *Die Abhängigkeit des Dirac-Operators von der Spin-Struktur*, Coll. Math. XLVII (57-62), 1984.

[**Fr2**] Friedrich T., *Dirac Operator in Riemannian Geometry*, Amer. Math. Soc. GSM **25**, 1997.

[**GLP**] Gilkey P.B., Leahy J.V., Park J., *Spectral Geometry, Riemannian submersions and the Gromov-Lawson conjecture*, Chapman & Hall, SAM 1999.

[**LM**] Lawson H.B., Michelsohn M.L., *Spin geometry*, Princeton University Press, NJ, 1989.

[**MR**] Miatello R., Rossetti J.P. *Flat manifolds isospectral on p-forms*, Jour. Geom. Anal. 11 (649-667), 2001.

[**MR2**] Miatello R., Rossetti J.P. *Comparison of twisted Laplace p-spectra for flat manifolds with diagonal holonomy*, Ann. Global Anal. Geom. 21 (341-376), 2002.

[**MR3**] Miatello R., Rossetti J.P. *P-spectrum and length spectrum of compact flat manifolds*, to appear Jour. Geom. Anal. Preprint, arXiv math.DG/0110325.

[**Pf**] Pfäffle F., *The Dirac spectrum of Bieberbach manifolds*, J. Geom. Phys. **35** (367–385), 2000.

[**Su**] Sunada T., *Riemannian coverings and isospectral manifolds*, Annals of Math. 121 (169-186), 1985.

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